**Planar Trusses:**

Loads: applied at the joints only

Internal Forces: axial tension or compressions

$$\sum F_x = 0 \quad \sum F_y = 0$$

Method of Joints: 2 Equilibrium Equations per joint

Method of Sections: 3 Equilibrium Equations per Section

Sign Convention: Assume that unknown internal forces are tensile (pulling away from the member)

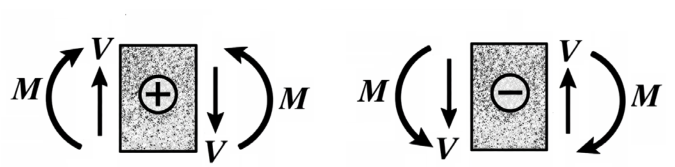
**Beams:**

Loads: Concentrated forces, couples, distributed loads

Internal Forces: shear forces and bending moments

Equilibrium: 3 equations for whole beam or section of beam (e.g. 2 force Eqs. & 1 Moment Eq. or 1 Force Eq. and 2 Moment Eqs. etc…)

Sign Convention: shear forces creating clockwise couple are positive bending moments creating compression at top of member are positive




Bending moment drawn on compression side of member.

**Planar Frames**

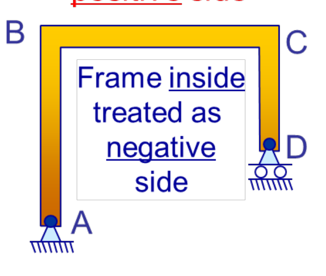
Loads: Concentrated forces, couples, distributed loads

Internal Forces: axial forces, shear forces and bending moments

Equilibrium: 3 equations for whole frame or section of frame (e.g. 2 force Eqs. & 1 Moment Eq. or 1 Force Eq. and 2 Moment Eqs. etc…)

Sign Convention: same as for beam with frame outside treated same as top of beam (positive side)

Frame ouside treated as positive side




Support Reactions & Displacements:

Fixed Support: 3 reactions & 0 displacement

Simple Support: 2 reactions & 1 displacement

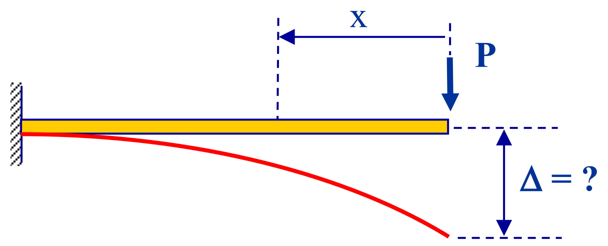
Roller Support: 1 reaction & 2 displacements

No Support: 0 reaction & 3 displacements

## Principle of Conservation of Energy

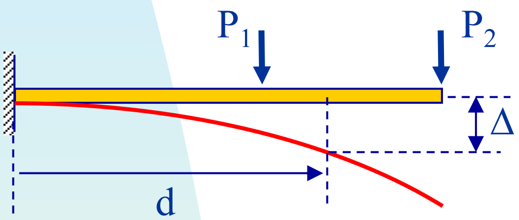
$U_e = U_i$

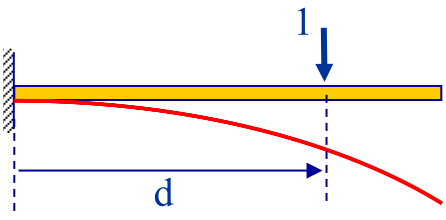
**Example:**




$$\textit{Beams and Frames}: \frac{1}{2}P\Delta = \int_0^L \frac{1}{2}\frac{M^2}{EI}dx \qquad \textit{Trusses}: \frac{1}{2}P\Delta = \sum_{i=1}^{n}\frac{N_i^2 L_i}{2A_iE_i}$$

## Unit Load Method (ULM)




**Beams and Frames**

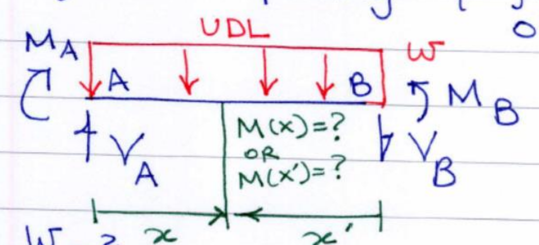
$$1.\Delta = \sum_{i=1}^{allmembers}\int_0^L m_i\frac{M_i}{E_iI_i}dx \qquad \textbf{or} \qquad 1.\theta = \sum_{i=1}^{allmembers}\int_0^L m_i\frac{M_i}{E_iI_i}dx$$

**Trusses** $1.\Delta = \sum_{i=1}^{all\,members}\frac{n_i.N_i.L_i}{A_iE_i}$

## Moment Equation for ULM

general Rule for determining Moment Equation needed for Computing $U_i = \int_0^L \frac{M^2}{2EI}dx$

or $\int_0^L \frac{mM}{EI}dx$




$M(x) = M_A + V_A x - \frac{w}{2}x^2$

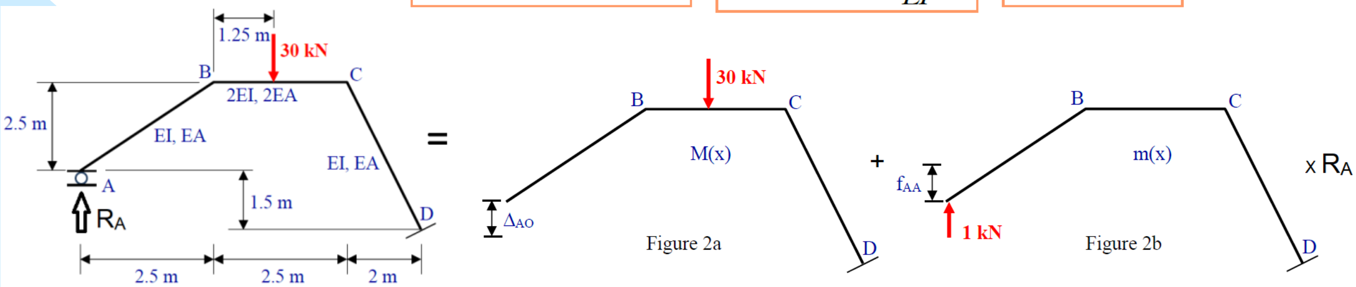
OR

$M(x') = M_B - V_B x - \frac{w}{2}x^2$

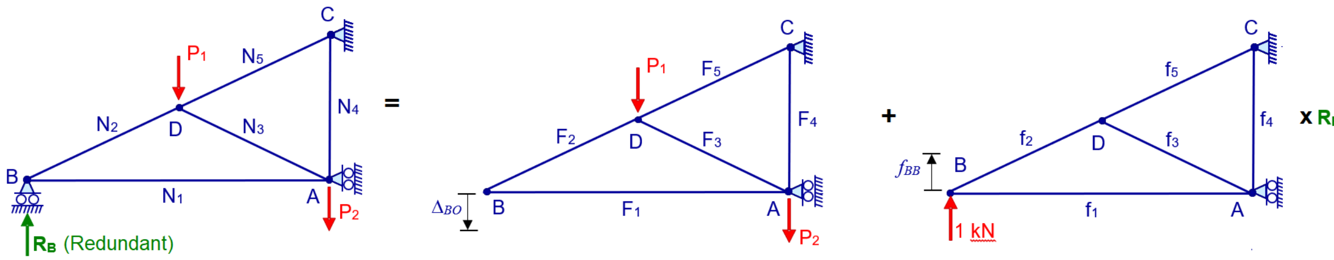
## Force Method

### Beams & Frames

$$a)\ \Delta_{AO} = \sum \int_0^L \frac{Mm}{EI} dx \qquad b)\ f_{AA} = \sum \int_0^L \frac{m^2}{EI} dx \qquad c)\ R_A = -\frac{\Delta_{AO}}{f_{AA}}$$




### Trusses



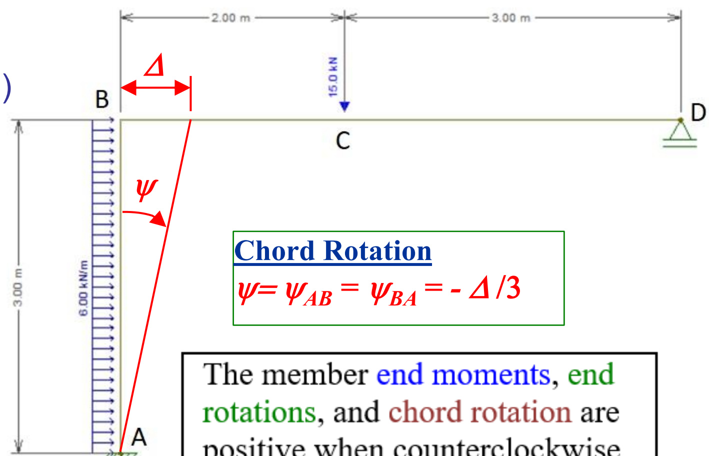

$$\Delta_{BO} = \sum \frac{f_i F_i L_i}{E_i A_i} \qquad f_{BB} = \sum \frac{f_i^2 L_i}{E_i A_i} \qquad R_B = -\frac{\Delta_{BO}}{f_{BB}}$$

## Slope Deflection Method

1. Determine No. of DOF (Unknown Displ.)
2. Compute Fixed-End Moments (FEM)
3. Write Slope-Deflection Equations

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - 3\psi\right) + FEM_{AB}$$
$$M_{BA} = \frac{2EI}{L}\left(\theta_A + 2\theta_B - 3\psi\right) + FEM_{BA}$$

4. Write Equilibrium Equations at the joints and Shear Equation when applicable (e.g. $M_{BA} + M_{BD} = 0$)
5. Solve for the unknow displacements (e.g. $\theta_B$, $\theta_D$, $\Delta$)
6. Compute the End Moments: $M_{AB}$, $M_{BA}$, etc…
7. Compute Shear Forces: $V_{AB}$, $V_{BA}$, etc…
8. Compute support reactions when needed
9. Draw Shear & Moments Diagrams when needed




**Chord Rotation**

$\psi = \psi_{AB} = \psi_{BA} = -\Delta/3$

The member end moments, end rotations, and chord rotation are positive when counterclockwise

*Frame sidesway degrees of freedom*

$$ss = 2j - \left[2(f+h) + r + m\right]$$

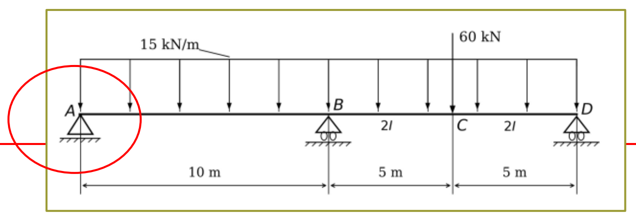
j = number of joints (3)
f = number of fixed supports (1)
h = number of hinged supports (0)
r = number of roller support (1)
m = number of (inextensible) members (2)

$$ss = 2(3) - \left[2(1+0) + 1 + 2\right] = 1$$

### Modified Slope-Deflection Method

$$M_{BA} = \frac{3EI}{L}\left(\theta_B - \psi\right) + \left(FEM_{BA} - \frac{FEM_{AB}}{2}\right)$$
$$M_{AB} = 0$$

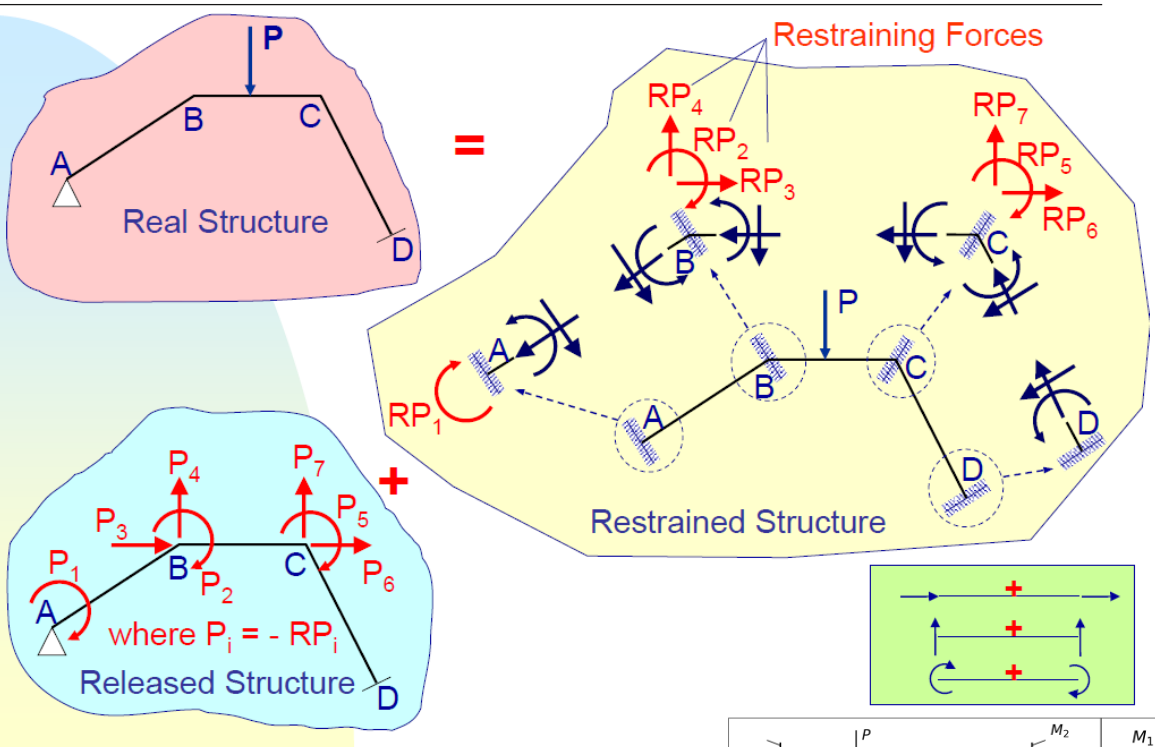
$$\theta_A = -\frac{\theta_B}{2} + \frac{3}{2}\psi - \frac{L}{4EI} FEM_{AB}$$

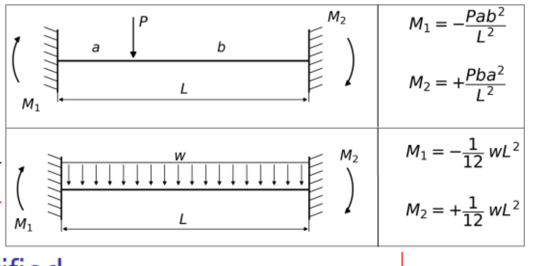



Prof. M. Maalej

# Stiffness Method

## Equivalent Joint Loads




## Method of Stiffness Matrix Analysis

**Method of Stiffness Matrix Analysis**

(a) Number of Degrees of Freedom is identified

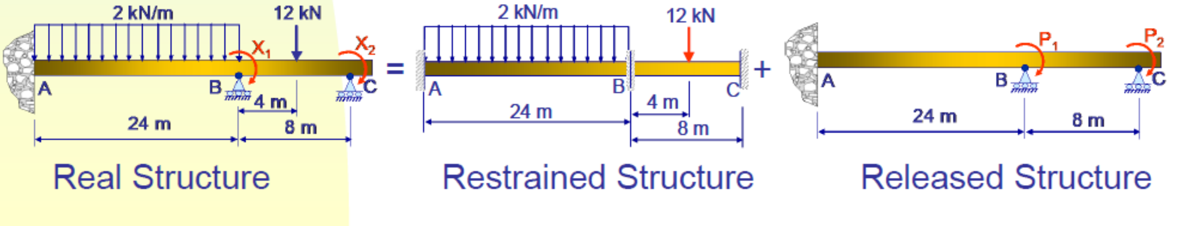
(b) Member internal forces in the **restrained structure** are obtained

(c) Equivalent joint loads are computed

**(d) Released Structure** (subjected to the Equivalent Loads) is analysed for joint displacements and member internal forces

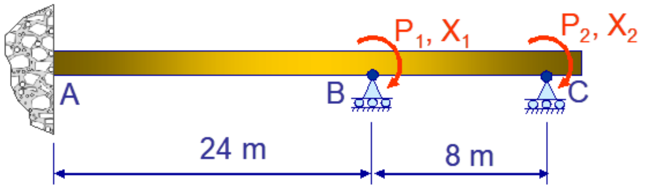
(e) Member internal forces in the **released state** [step (d)] are added to Member internal forces in the **restrained state** [step (b)] to obtain the Member internal forces in the **real structure**.

(f) Joint Displacement of the **real structure** are the same as those in the **released structure** [step (d)]



## Stiffness Equations at Structure Level

Joint Loads-Joint Displacements Relationships (Released State)




Released Structure

**K X = P**

In the released structure, the joint loads $P_1$-$P_2$ are related to the joint rotations $X_1$-$X_2$ by the following relationship:

$$\begin{cases} P_1 = K_{11}X_1 + K_{12}X_2 \\ P_2 = K_{21}X_1 + K_{22}X_2 \end{cases}$$

The above relationship can also be expressed in matrix form **P = K.X** where **K** is referred to as the **Structure Stiffness Matrix**

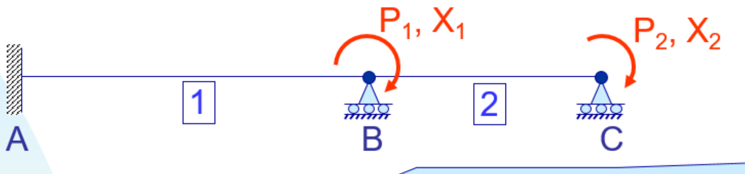
$$\underbrace{\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}}_{\mathbf{P}} = \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{\mathbf{X}}$$

The "$K_{ij}$" of the matrix K are referred to as the stiffness coefficients.

## Stiffness Equations at Member Level

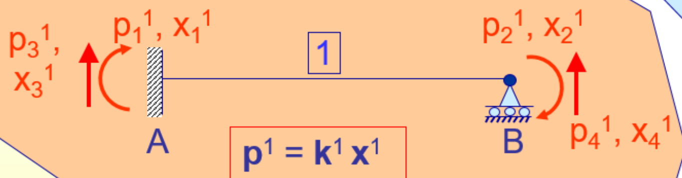
Member End Forces-End Displacements Relationships (Released State)

Structure Level

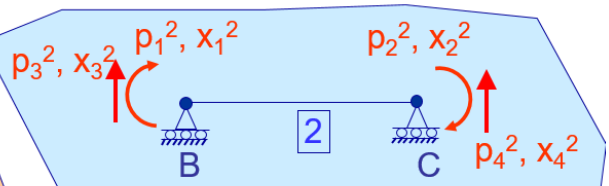



Member Level

$$\mathbf{x}^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{bmatrix} = \begin{bmatrix} 0 \\ X_1 \\ 0 \\ 0 \end{bmatrix}$$




$\mathbf{p}^1 = \mathbf{k}^1\,\mathbf{x}^1$




$\mathbf{p}^2 = \mathbf{k}^2\,\mathbf{x}^2$

$$\mathbf{x}^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ 0 \\ 0 \end{bmatrix}$$

# Beam Member Stiffness Matrix

$$
k^m = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\
\begin{bmatrix}
\frac{4EI}{L} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{6EI}{L^2} \\
\frac{2EI}{L} & \frac{4EI}{L} & \frac{-6EI}{L^2} & \frac{6EI}{L^2} \\
\frac{-6EI}{L^2} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \frac{-12EI}{L^3} \\
\frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & \frac{12EI}{L^3}
\end{bmatrix}
\end{array}
$$


## $k^1$ for member 1

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EI = L =

## $k^2$ for member 2

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EI = L =

## $k^3$ for member 3

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EI = L =

## $k^4$ for member 4

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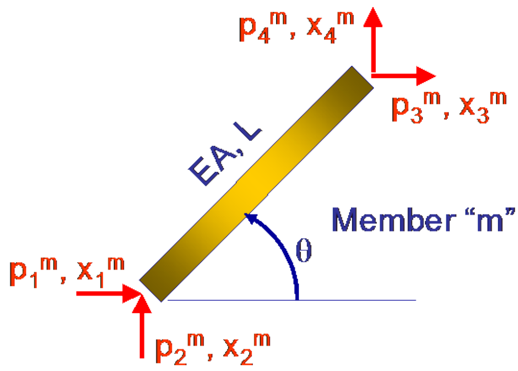
EI = L =

## Truss Member Stiffness Matrix

### $k^m$ for member m

$$
\begin{array}{c|c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\begin{bmatrix}
\frac{EA}{L}\cos^2\theta & \left(\frac{EA}{L}\right)\sin\theta\cos\theta & -\frac{EA}{L}\cos^2\theta & \left(-\frac{EA}{L}\right)\sin\theta\cos\theta \\
\left(\frac{EA}{L}\right)\sin\theta\cos\theta & \frac{EA}{L}\sin^2\theta & \left(-\frac{EA}{L}\right)\sin\theta\cos\theta & -\frac{EA}{L}\sin^2\theta \\
-\frac{EA}{L}\cos^2\theta & \left(-\frac{EA}{L}\right)\sin\theta\cos\theta & \frac{EA}{L}\cos^2\theta & \left(\frac{EA}{L}\right)\sin\theta\cos\theta \\
\left(-\frac{EA}{L}\right)\sin\theta\cos\theta & -\frac{EA}{L}\sin^2\theta & \left(\frac{EA}{L}\right)\sin\theta\cos\theta & \frac{EA}{L}\sin^2\theta
\end{bmatrix}
\end{array}
$$




### $k^1$ for member 1

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EA = L = EA/L =
cos θ = sin θ = cos θ sin θ =
$\cos^2\theta$ = $\sin^2\theta$ =

### $k^2$ for member 2

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EA = L = EA/L =
cos θ = sin θ = cos θ sin θ =
$\cos^2\theta$ = $\sin^2\theta$ =

### $k^3$ for member 3

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EA = L = EA/L =
cos θ = sin θ = cos θ sin θ =
$\cos^2\theta$ = $\sin^2\theta$ =

### $k^4$ for member 4

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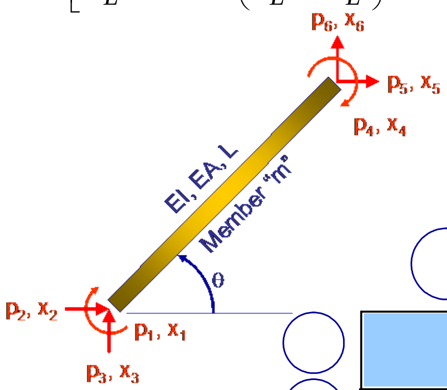
EA = L = EA/L =
cos θ = sin θ = cos θ sin θ =
$\cos^2\theta$ = $\sin^2\theta$ =

## Frame Member Stiffness Matrix

$$
\mathbf{k}^m = \begin{bmatrix}
\boxed{\frac{4EI}{L}} & \frac{6EI}{L^2}\sin\theta & \frac{-6EI}{L^2}\cos\theta & \frac{2EI}{L} & \frac{-6EI}{L^2}\sin\theta & \frac{6EI}{L^2}\cos\theta \\
\frac{6EI}{L^2}\sin\theta & \boxed{\frac{12EI}{L^3}\sin^2\theta+\frac{EA}{L}\cos^2\theta} & \left(\frac{-12EI}{L^3}+\frac{EA}{L}\right)\sin\theta\cos\theta & \frac{6EI}{L^2}\sin\theta & -\frac{12EI}{L^3}\sin^2\theta-\frac{EA}{L}\cos^2\theta & \left(\frac{12EI}{L^3}-\frac{EA}{L}\right)\sin\theta\cos\theta \\
\frac{-6EI}{L^2}\cos\theta & \left(\frac{-12EI}{L^3}+\frac{EA}{L}\right)\sin\theta\cos\theta & \boxed{\frac{12EI}{L^3}\cos^2\theta+\frac{EA}{L}\sin^2\theta} & -\frac{6EI}{L^2}\cos\theta & \left(\frac{12EI}{L^3}-\frac{EA}{L}\right)\sin\theta\cos\theta & -\frac{12EI}{L^3}\cos^2\theta-\frac{EA}{L}\sin^2\theta \\
\frac{2EI}{L} & \frac{6EI}{L^2}\sin\theta & -\frac{6EI}{L^2}\cos\theta & \boxed{\frac{4EI}{L}} & \frac{-6EI}{L^2}\sin\theta & \frac{6EI}{L^2}\cos\theta \\
\frac{-6EI}{L^2}\sin\theta & -\frac{12EI}{L^3}\sin^2\theta-\frac{EA}{L}\cos^2\theta & \left(\frac{12EI}{L^3}-\frac{EA}{L}\right)\sin\theta\cos\theta & \frac{-6EI}{L^2}\sin\theta & \boxed{\frac{12EI}{L^3}\sin^2\theta+\frac{EA}{L}\cos^2\theta} & \left(-\frac{12EI}{L^3}+\frac{EA}{L}\right)\sin\theta\cos\theta \\
\frac{6EI}{L^2}\cos\theta & \left(\frac{12EI}{L^3}-\frac{EA}{L}\right)\sin\theta\cos\theta & -\frac{12EI}{L^3}\cos^2\theta-\frac{EA}{L}\sin^2\theta & \frac{6EI}{L^2}\cos\theta & \left(-\frac{12EI}{L^3}+\frac{EA}{L}\right)\sin\theta\cos\theta & \boxed{\frac{12EI}{L^3}\cos^2\theta+\frac{EA}{L}\sin^2\theta}
\end{bmatrix}
$$




$$\cos\theta = C \qquad \sin\theta = S$$

$\mathbf{k}^1$

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$EI =$ $EA =$ $L =$ $EA/L =$

$4EI/L =$ $2EI/L =$ $6EI/L^2 =$ $12EI/L^3 =$

$C =$ $S =$ $C^2 =$ $S^2 =$ $SC =$

$\mathbf{k}^2$

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$EI =$ $EA =$ $L =$ $EA/L =$

$4EI/L =$ $2EI/L =$ $6EI/L^2 =$ $12EI/L^3 =$

$C =$ $S =$ $C^2 =$ $S^2 =$ $SC =$